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# PROBLEMS ON CHAIN PARTITIONS

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## Problems on chain partitions

Recently I have come across several fundamental problems concerning the existence of partitions of posets into chains satisfying certain conditions. Other problems of this sort have been open and frustrating researchers for years. I have collected these problems here, together with a few of my own, to give them wider exposure. The problems can be easily stated for a broad mathematical audience. It is likely that their solutions require techniques and insights which differ greatly from those normally employed in the contexts in which the problems were initially developed.

The presentation of the problems follows a review of the necessary notation and terminology. We first discuss problems about the poset of subsets of a finite set, and then we give problems for other families of posets.

### *Notation and terminology*

Throughout we consider finite posets $P = (P, \leqslant)$. A totally ordered subset of $P$, say $C = \{x_1, \ldots, x_r\} \subseteq P$ with $x_1 < x_2 < \cdots < x_r$, is called a *chain*. Such a chain is *saturated* (also called unrefinable or consecutive) if for all $i \geqslant 2$, $x_{i+1}$ *covers* $x_i$, that is, $x_i \leqslant y < x_{i+1}$ for $y$ in $P$ only if $y = x_i$. An *antichain* is a totally unordered subset of $P$. The *width* of $P$, denoted $d_1(P)$, is the maximum size of an antichain in $P$. Let $A$ be a maximum-sized antichain, and let $\boldsymbol{C} = \{C_1, \ldots, C_s\}$ be a partition of $P$ into chains $C_i$. Since any chain $C_i$ intersects the antichain $A$ at most once, it follows that

$$|\boldsymbol{C}| = s \geqslant |A| = d_1(P).$$

A theorem of Dilworth [5] states that this bound on the number of chains in a chain partition is best-possible: There exists $\boldsymbol{C}$ such that $|\boldsymbol{C}| = d_1(P)$.

We next recall an important generalization of Dilworth's Theorem. Given $k \geqslant 1$, a subset $F$ of $P$ is a *$k$-family* if it can be expressed as the union of at most $k$

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antichains (or, equivalently, if $|F \cap C| \leq k$ for every chain $C \subseteq P$). Let $d_k(P) = \max\{|F| : F \subseteq P$ a $k$-family$\}$. A chain partition $\boldsymbol{C} = \{C_1, \ldots, C_s\}$ induces an upper bound on the size of a $k$-family $F$ in the following way:

$$|F| = \sum_i |F \cap C_i| \leq \sum_i \min(k, |C_i|).$$

We denote this last sum by $d(k, \boldsymbol{C})$. By taking $F$ to be a maximum-sized $k$-family, we obtain $d(k, \boldsymbol{C}) \geq d_k(P)$. If $d(k, \boldsymbol{C}) = d_k(P)$, $\boldsymbol{C}$ is said to be $k$-saturated. The Greene–Kleitman theorem [8] states that for all $k$ and $P$ there exists a $k$-saturated chain partition $\boldsymbol{C}$. Dilworth's Theorem is the case $k = 1$.

All posets $P$ we consider are *graded* which means that every maximal chain $C$ in $P$ has the same size. The *rank* $r(x)$ of an element $x$ in a graded poset $P$ is one less than the maximum size of the chains which contain $x$ as the top element. The *rank of $P$, $r(P)$,* is the maximum rank of any element in $P$. Let $P_i = \{x \in P : r(x) = i\}$. This partitions $P$ into antichains, $P_0, \ldots, P_n$, where $n = r(P)$.

The sequence of Whitney numbers of a graded poset $P$ is $(W_0, W_1, \ldots, W_n)$ where $W_i = |P_i|$ and $n = r(P)$. $P$ is *rank-symmetric* if $W_i = W_{n-i}$ for all $i$, and *rank-unimodal* if for some $j$, $W_0 \leq W_1 \leq \cdots \leq W_j$ and $W_j \geq W_{j+1} \geq \cdots \geq W_n$.

The union of the $k$ largest rank-sets $P_i$ of $P$ is a $k$-family so $d_k(P)$ is at least the sum of the $k$ largest Whitney numbers $W_i$. $P$ has the *strong Sperner property* if for all $k$, $d_k(P)$ actually equals this sum of the $k$ largest Whitney numbers. A *Peck poset* is a rank-symmetric, rank-unimodal poset with the strong Sperner property (cf. survey [13]).

If a graded poset $P$ of rank $n$ has a partition $\boldsymbol{C}$ into chains such that each chain in $\boldsymbol{C}$ is saturated and symmetric about middle rank, $\frac{1}{2}n$, then $P$ is called a *symmetric chain order*. This means that for each chain $C$ in $\boldsymbol{C}$ there exists an $i$ such that $C$ consists of one element of each rank $i, i+1, \ldots, n-i$. Computing the bound $d(k, \boldsymbol{C})$, we find that $d(k, \boldsymbol{C}) = d_k(P)$ is the sum of the $k$ middle Whitney numbers, so that $P$ is a Peck poset. We also conclude that the partition $\boldsymbol{C}$ is *completely saturated,* which means it is $k$-saturated for all $k$.

*Problems concerning subsets*

Let $[n] = \{1, \ldots, n\}$. Let $B_n = (2^{[n]}, \subseteq)$ be the Boolean algebra of order $n$, which is the poset of all subsets of $[n]$, ordered by inclusion. Sperner's Theorem [22] states that $B_n$ has width $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$, the size of the middle rank(s). A stronger result, due to deBruijn et al. [4], is that $B_n$ is a symmetric chain order (cf. [10, 11]).

*1 (Z. Füredi [7]).* Can $B_n$ be partitioned into $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$ chains of the same size (within one)? That is, letting $a$ and $b$ satisfy $2^n = a\binom{n}{\lfloor \frac{1}{2}n \rfloor} + b$, where $0 \leq b < \binom{n}{\lfloor \frac{1}{2}n \rfloor}$, we require $b$ chains of size $a+1$ and $\binom{n}{\lfloor \frac{1}{2}n \rfloor} - b$ of size $a$. This is the minimum number of chains which can cover $B_n$, in light of Sperner's Theorem. This problem seems to be hard.

*2 (B. Sands [21]).* Can $B_n$ be partitioned into chains of size 4 for sufficiently large $n$? More generally, for any $k$, can $B_n$ be partitioned into chains of size $2^k$ for all sufficiently large $n$, given $k$?

In this question note that $2^k$ divides $|B_n| = 2^n$ for $n \geq k$, so a complete partition is possible. For $k = 1$ it is trivial to partition $B_n$ for $n \geq 1$, e.g. take the chains $\{X, X \cup \{n\}\}$ for $X \subseteq [n-1]$. For $k = 2$, Griggs et al. [26] recently solved the original problem by showing that $B_n$ can be partitioned into chains of size 4 if and only if $n \geq 9$. It is impossible for $n \leq 8$ since then the number of chains, $2^{n-2}$, is less than $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$. For $n = 9$ they have an ad hoc construction, which can be extended automatically by induction to all $n > 9$. However, the method does not extend to the general $k$ problem. Of course it would be very nice if $B_n$ could be partitioned into chains of size $2^k$ if and only if the number of chains $2^{n-k} \leq \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

We propose a stronger conjecture which involves chain sizes other than powers of 2. Given $c \geq 1$, can $B_n$ be partitioned into chains of size $c$, except for at most $c - 1$ elements, which also belong to a single chain, for $n > n_0(c)$? This is trivial for $c = 1$ and is true for $c = 2, 4$ by the remarks above. For $c = 3$ it turns out to be true for all $n$.

*3.* Problems 1 and 2 seem promising particularly because Sperner theory is consitent with them. The strongest conjecture about chain sizes in partitions of $B_n$ that is consistent with Sperner theory is this: Let $\boldsymbol{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be any partition of the integer $2^n$ into parts $\lambda_i \geq 0$. Let $\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots)$ be the partition of $2^n$ corresponding to the sizes of the chains in the symmetric chain decomposition of $B_n$. Thus,

$$\sigma_1 = n + 1, \qquad \sigma_2 = \cdots = \sigma_{\binom{n}{1}} = n - 1, \qquad \sigma_{\binom{n}{1}+1} = \cdots = \sigma_{\binom{n}{2}} = n - 3,$$

and so on. A term $\sigma_i > 0$ if and only if $i \leq \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

The conjecture is that there is a partition of $B_n$ with chain sizes $\boldsymbol{\lambda}$ if and only if $\boldsymbol{\sigma} \geq \boldsymbol{\lambda}$ in the majorization order, that is, $\sum \sigma_i = \sum \lambda_i$ and for all $j$

$$\sum_{i=1}^{j} \sigma_i \geq \sum_{i=1}^{j} \lambda_i.$$

The "only if" direction is a consequence of the symmetric chain decomposition being completely saturated. We have only checked the converse direction for very small cases, $n \leq 4$. The analogous conjecture for arbitrary posets is easily seen to be false, so this conjecture may be overly optimistic.

*4.* Determine partitions $\boldsymbol{\lambda}$ of $2^n$ such that $B_n$ can be partitioned into *saturated* chains with sizes $\boldsymbol{\lambda}$. The symmetric chain decomposition described by $\boldsymbol{\sigma}$ above consists of saturated chains, but the conjecture for *saturated* chains analogous to problem 3 is false: There exists $\boldsymbol{\lambda} \leq \boldsymbol{\sigma}$ not realizable by a partition of $B_n$ into saturated chains.

We recently obtained a partial answer to this problem [14]: Let $G(n, c, k) = \sum\{\binom{n}{i} : i \equiv k \pmod{c}\}$. If $B_n$ is partitioned into saturated chains of size at most $c$ (i.e. if $\lambda_1 \leqslant c$), then we show that

(i) the number of chains of size $c$ is at most $G(n, c, \lfloor\frac{1}{2}(n+c)\rfloor)$ (i.e. $\lambda_i < c$ for $i > G(n, c, \lfloor\frac{1}{2}(n+c)\rfloor)$, and

(ii) the total number of chains is at least $G(n, c, \lfloor\frac{1}{2}n\rfloor)$ (i.e. $\lambda_i > 0$ for $i = G(n, c, \lfloor\frac{1}{2}n\rfloor)$).

Obtaining these bounds is easy, but it is interesting that both bounds are best-possible.

*Problems for other posets*

*5 (R. Stanley [23, p. 182]).* Let $L(m, n)$ denote the lattice of Ferrers diagrams fitting into an $m \times n$ rectangle, ordered by inclusion. Equivalently, $L(m, n)$ is the poset of all integer sequences $S = (0 \leqslant a_1 \leqslant \cdots \leqslant a_m \leqslant n)$ ordered by $S \leqslant S'$ if and only if $a_i \leqslant a_i'$ for $1 \leqslant i \leqslant m$. Is $L(m, n)$ a symmetric chain order? This was first shown for $m \leqslant 4$ and all $n$ by Riess [20] and later rediscovered for $m = 3$ by Lindström [17] and for $m = 4$ by West [25]. Stanley proved in general that $L(m, n)$ is Peck (cf. [24, 15, 19] for later, more elementary proofs). Stanley [23] studied more generally a class of posets related to the Bruhat order of Weyl groups. He showed that these posets are Peck and asked whether they are symmetric chain orders.

*6 (A. Björner [3, p. 189]).* Is the weak ordering of the symmetric group, $S_n$, a symmetric chain order? In this ordering, a permutation $\boldsymbol{a} = (a_1, \ldots, a_n)$ covers $b = (b_1, \ldots, b_n)$ if $\boldsymbol{a}$ is obtained from $\boldsymbol{b}$ by transposing some adjacent pair $b_i, b_{i+1}$ in $\boldsymbol{b}$ with $b_i < b_{i+1}$. For instance $(1, 4, 2, 3)$ covers $(1, 2, 4, 3)$ in $S_4$. The minimum element is thus $(1, 2, \ldots, n)$, the maximum element is $(n, n-1, n-2, \ldots, 1)$, and the rank of this ordering of $S_n$ is $\frac{1}{2}(n(n-1))$. This ordering is rank-symmetric and rank-unimodal, but it is open whether it even has the Sperner property. The same questions are posed more generally for the weak ordering of a Coxeter group $(W, S)$ with $W$ finite [3].

*7 (Folklore).* Let $L_n(q)$ denote the lattice of subspaces of an $n$-dimensional vector space $V$ over the finite field $\mathrm{GF}(q)$, ordered by inclusion. Then a subspace $S$ of $V$ has rank equal to its dimension. $L_n(q)$ is known to be a symmetric chain order [1, 12] by an existence proof which exploits the regularity of the lattice and combinatorial matching theory. The problem is to give an *explicit* symmetric chain decomposition analogous to the explicit ones for the Boolean lattice and the lattice of divisors of an integer. Such a result may require a nice method of describing the subspaces which would be helpful for other problems about $L_n(q)$, e.g. proving a subspace analogue of the Kruskal–Katona theorem.

*8 (K. Engel [6]).* An ordered partition $\boldsymbol{C}$ of a poset $P$ into chains $C_1, \ldots, C_t$ is said to be *admissible* if $|C_1| \leq 3$ and if whenever two elements $x$ and $y$ belong to $C_j$ and $C_j$ has at least two elements above $x$ and below $y$, there is some element $z$ in $C_1 \cup \cdots \cup C_{j-1}$ such that $x < z < y$. The problem is to show that the product of three chains with equal length has an admissible ordered symmetric chain decomposition. This poset is known to be a symmetric chain order [4], so the problem is to find a decomposition which can be ordered appropriately. Engel found an admissible decomposition for the product of just two chains of equal length using a lovely zigzag construction. This result extends to every product of an *even* number of equal length chains by a product theorem for admissible partitions. To extend this to *any* product of at least two chains of equal length it suffices, by the product theorem, to solve the case of just three chains. This cannot follow simply from the product theorem since a single chain of size at least 4 is not admissible. Supporting evidence for the conjecture is the discovery by Mahnke [18] of admissible decompositions when the chain size is at most 5.

If the desired result is true, there would be a nice application to computing the minimum number of evaluations required to completely determine an unknown order-preserving map $f: P \to Q$ when $P$, $Q$ are finite posets and $P$ is a product of chains of equal length.

*9 (Griggs [12]).* A finite ranked poset $P$ has the *LYM property* if for all $k > 0$ and all subsets $A \subseteq P_k$, the set $\partial A$ of elements of $P_{k-1}$ covered by some element of $A$ satisfies

$$\frac{|\partial A|}{|P_{k-1}|} \geq \frac{|A|}{|P_k|}.$$

One class of LYM posets is the *regular* posets, which have the property that for all $k$, every element of $P_k$ covers the same number $\alpha_k$ of elements of $P_{k-1}$ and is covered by the same number $\beta_k$ of elements of $P_{k+1}$. Anderson [1] and, independently, Griggs [12] proved that every rank-symmetric, rank-unimodal LYM poset is a symmetric chain order. The problem is to say something about LYM posets in general. Specifically, Griggs has conjectured since 1975 that LYM posets have completely saturated partitions. Since LYM posets $P$ are known to be strong Sperner [10], this says equivalently that there is a partition $\boldsymbol{C}$ of $P$ into chains such that whenever a chain $C$ in $\boldsymbol{C}$ contains an element of any rank $P_i$ it also contains an element of each rank $P_j$ such that $|P_j| \geq |P_i|$. It might be easier to prove a considerably weaker result, e.g. that a rank-unimodal regular poset has such a completely saturated chain partition. However, no progress has been made on this problem since its formulation.

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